Simple hierarchical systems: Stability, self-organized criticality, and catastrophic behavior

E. M. Blanter and M. G. Shnirman
International Institute of Earthquakes Prediction Theory and Mathematical Geophysics, Varshavskoe sh. 79/2, Moscow 113556, Russia

(Received 5 November 1996)

A description of various kinds of behavior for a hierarchical model of defect development, representing a transition from stability to catastrophe, is suggested. It is shown that the self-organized criticality regarded as a linear form of the magnitude-frequency relation on a wide area of parameter corresponds to different kinds of system behavior. Examples of the systems representing self-organized criticality with stationary, periodic, and chaotic relation between the density of defects and the level of hierarchy are suggested. A complex behavior, when areas of self-organized criticality alternate with areas of catastrophe and/or stability, is observed in the model. Examples considered in the paper perform basic kinds of possible behavior on the transition interval from stability to catastrophe for a simple class of hierarchical systems. [S1063-651X(97)05404-4]

PACS number(s): 64.60.Lx, 05.45.+b, 05.40.+j

I. INTRODUCTION

Hierarchical systems of defects development are widely used in the modeling of destruction experiments [1,2], areas of sources of big earthquakes [3], areas of triple junctions [4], and global seismicity [5–10]. The defects in hierarchical systems are referred to as cracks for modeling of destruction experiments or as earthquakes in seismic models. The size of defects is related with a level of hierarchy and corresponds to the size of cracks or source area of earthquakes. Hierarchical structure represents multiscale properties of real systems that are reflected in the power-law form of certain basic relations [11].

The linear behavior of the magnitude-frequency relation in a log/log plot, observed for the world seismicity [12], has been recently explained as self-organized criticality [10,11,14,16]. The self-organized criticality phenomenon was observed in several seismic models such as the avalanche models [13,14,10], uniform Burridge and Knopoff model [17], and the model of Olami *et al.* [18–20]. To generalize different observations in real and model systems, the self-organized criticality may be understood as a linear form of the magnitude-frequency relation on the log/log plot for a large area of system parameters. In the present work we use the notion of self-organized criticality in this sense. Recently the self-organized criticality was demonstrated by various hierarchical systems: hierarchical models with feedback [6,9] and dynamic hierarchical models with pattern healing [8,21].

A phase transition from stability to catastrophe was observed in some hierarchical models [1,6] and applied to the description of the behavior of cracks in laboratory experiments of sample destruction. In the phase of stability the number of cracks exponentially decreases when the size of the cracks grows. There is no sufficiently large cracks in this case. The catastrophic behavior is characterized by an increase of the density of cracks with linear size and the appearance of a global fracture that corresponds to the total destruction of a sample. Thus there are three kinds of possible behavior observed in hierarchical systems: stability, self-organized criticality, and catastrophe. A phase transition from stability to self-organized criticality was observed in a dynamic hierarchical model [8,21]. In the present paper an investigation of self-organized criticality appearing inside the transition area between areas of stability and catastrophe in a simple one-parametrical hierarchical model is suggested.

A general description of the model is suggested in Secs. II–V. A phase transition from stability to catastrophe in terms of the model considered is described in Sec. VI. Hierarchical systems presented in Sec. VII have a transition interval of the self-organized criticality between the areas of stability and catastrophe. Examples of different kinds of relationships between the density of defects and the corresponding level are suggested in this section. These examples correspond to a linear form of the magnitude-frequency relation for all values of the parameter inside the transition interval. A more complicated behavior demonstrating the intermittency of intervals of self-organized criticality with intervals of catastrophe is suggested in Sec. VIII. The alternation of stability and self-organized criticality areas is described in Sec. IX. Results and possible analogies are discussed in Sec. X.

II. GENERAL DESCRIPTION OF THE MODEL

We consider a hierarchical system with L levels and a branch number n (Fig. 1). The first level is the lowest in the system. Each element of the upper level $l+1$ is composed of n elements of the previous level l. Elements of level l composing one element of the upper level $l+1$ are referred to as a group. There are two possible states of elements of the system. One of them is referred to as a defect state. The state of an element of the level $l+1$ depends on the configuration of defects in the corresponding group of elements of the previous level l. Some configurations of defects in the group are specified as critical configurations. If there is a critical configuration of defects in a group of elements of level l, then the corresponding element of the upper level $l+1$ is in the defect state. There are no defects without the corresponding critical configuration below, except that of the first level of the system. Thus, the states of all elements of the system are determined by the state of the first level. It is assumed that all elements on each level are independent of one an-

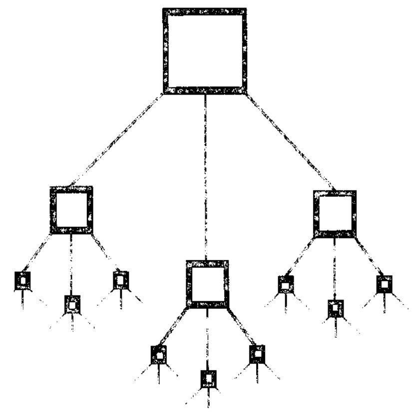

FIG. 1. Example of an hierarchical system with branch number 3. A group of 3 elements of the previous level composes an element of the next level.

other and have the same probability to be a defect p_l. Then densities of defects p_l for all levels are determined by the density of defects p_1 of the first level. The density of defects of the initial level $p=p_1$ is the single parameter of the system.

The probability of a configuration containing k defects at level l is equal to $p_l^k(1-p_l)^{n-k}$. The density of all critical configurations at level l is expressed as follows:

$$F(p_l)=\sum_{k=1}^{n} A_k p_l^k (1-p_l)^{n-k}, \tag{1}$$

where A_k is the number of critical configurations containing k defects. The density of defects of the upper level $l+1$ is equal to the density of critical configurations on the previous level l; then,

$$p_{l+1}=F(p_l). \tag{2}$$

The form of the transition function F is independent of the level of the system [see Eq. (1)]. The behavior of the densities of defects p_l is determined by the properties of the transition function F on the interval (0,1) and the density of defects p on the first level of the system.

III. MAGNITUDE-FREQUENCY RELATION

The magnitude-frequency relation is often applied to describe the behavior of complex systems of different nature. In the present work we use the magnitude-frequency relation to separate the main kinds of behavior of the system. Let us define the magnitude-frequency relation for a given class of hierarchical systems. In the model described above the total number of elements of the level l exponentially decrease when l grows:

$$N(l)=Cn^{L-l}, \tag{3}$$

where L is the total number of levels, C is the total number of elements on the highest level of the system. If the density of defects on the lth level is equal to p_l, then the average number of defects of this level is as follows:

$$\langle N_l\rangle = N(l)p_l. \tag{4}$$

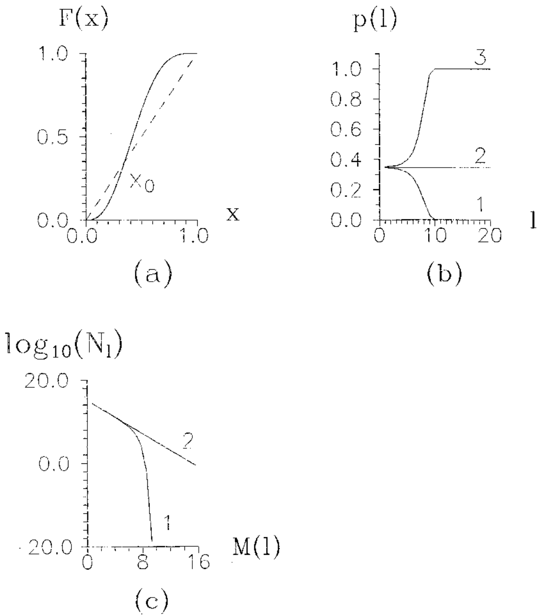

FIG. 2. Hierarchical system representing a phase transition from stability to catastrophe. Branch number of the model is equal to 6. All configurations including more than 2 defects are critical. (a) Transition function $F(x)$. The area of stability corresponds to the interval $(0,x_0)$, the area of catastrophe corresponds to the interval $(x_0,1)$. Unstable fixed point x_0 is a point of the phase transition from stability to catastrophe. (b) Densities of defects vs level of hierarchy for different values of parameter p. (1): in the area of stability $p=0.347<x_0$; (2): in the point of phase transition $p=x_0\approx 0.3471289$; (3): in the area of catastrophe $p=0.35>x_0$. (c) Magnitude-frequency relation for different values of parameter p. (1): in the area of stability $p<x_0$; (2): in the point of phase transition $p=x_0$. A slope of the straight line (2) is equal to unity.

All elements of the same level have the same size, which exponentially depends on the level. If the size of the elements of the first level is taken as unity then the size of the elements of level l is

$$S(l)=n^l. \tag{5}$$

In seismic investigations the magnitude of an earthquake is assumed to be proportional to the logarithm of the size of the source area. Similarly the magnitude of the defect of level l in the model is defined as follows:

$$M(l)=\log_{10}S(l)=l\log_{10}(n). \tag{6}$$

The magnitude-frequency relation describes a relation between the number of defects N_l and the magnitude $M(l)$. From Eqs. (3)–(6) it follows that

$$\log_{10}N_l = L\log_{10}(n) - M(l) + \log_{10}p_l + \log_{10}C. \tag{7}$$

IV. THE AREA OF STABILITY

Let us consider the transition function $F(x)$ defined by Eq. (1) in the interval (0,1). If configurations containing only one defect are not critical [$A_1=0$ in Eq. (1)], then the fixed point 0 [$F(0)=0$] is stable [Fig. 2(a)]. This means that if p is close to zero, than the sequence of densities of defects

p_l tends to zero when l grows [Fig. 2(b), curve (1)]. The corresponding magnitude-frequency relation has a double exponential downward bend [Fig. 2(c), curve (1)]. This kind of behavior is referred to as stability because there are no defects on high levels of the system.

V. THE AREA OF CATASTROPHE

Hierarchical models described in previous works [1,6] exhibit a phase transition from stability to catastrophe in the point $p=p_{\mathrm{cr}}$. The parametrical area $p>p_{\mathrm{cr}}$ corresponds to a catastrophic behavior of the system. If $p>p_{\mathrm{cr}}$ then the densities p_l of defects of the level l increase when l grows and tend to unity for the highest levels of the system [Fig. 2(b), curve (3)]. The unity value of densities of defects means a complete destruction of the corresponding levels of the system. If the transition function F has a stable fixed point 1 [$F(1)=1$], then for all values of parameter p taken in a neighborhood of unity, densities of defects p_l tend to unity when l grows. The condition $F(1)=1$ obviously means that the coefficient A_n in Eq. (1) is nonzero — $A_n>0$. There is a single configuration that includes n defects in a group of n elements, so $A_n=1$. The fixed point $F(1)=1$ is stable, if $F(x)-x>0$ when x is close to unity [Fig. 2(a)]. If $(1-x)$ is close to zero, then using the second order of approximation, we obtain the following condition for the coefficient A_{n-1}:

$$A_{n-1}>n-1. \tag{8}$$

There is only n configurations including $n-1$ defects in a group of n elements, so the catastrophe exists in a neighborhood of unity if $A_{n-1}=n$.

VI. A PHASE TRANSITION FROM STABILITY TO CATASTROPHE

Let us consider a hierarchical system with branch number equal to 6. All configurations containing more than 2 defects are critical. The transition function F is as follows:

$$F(x)=20x^3(1-x)^3+15x^4(1-x)^2+6x^5(1-x)+x^6. \tag{9}$$

The plot of the transition function is performed in Fig. 2(a). The function $F(x)$ has 3 fixed points. Two points — 0 and 1 — are stable; fixed point x_0 is unstable.

The phase transition from stability to catastrophe occurs in the point x_0 [Fig. 2(b)]. If the density of defects on the initial level p is less than x_0 the system demonstrates stability; if the parameter $p>x_0$ [Fig. 2(b), curve (1)], then it demonstrates catastrophic behavior [Fig. 2(b), curve (3)].

The magnitude-frequency relation is linear in the point $p=x_0$ [Fig. 2(c), curve (2)] and has a double exponential downward fall when $p<x_0$ [Fig. 2(c), curve (1)]. The similar phase transition from stability to catastrophe was previously observed in hierarchical models [1,6,7].

VII. SELF-ORGANIZED CRITICALITY AREA

The hierarchical system described above has a transition function F with 3 fixed points. The transition functions of systems considered below have 5 fixed points. To obtain areas of stability and catastrophe in the neighborhood of zero and unity, respectively, the transition function has to have 0 and 1 as stable fixed points.

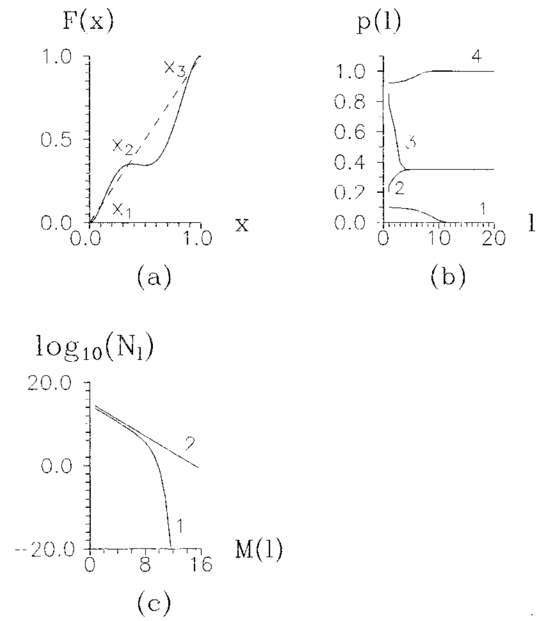

FIG. 3. Hierarchical system representing a stable behavior in the area of self-organized criticality. Branch number of the model is equal to 6. All configurations containing 2, 5, or 6 defects are critical. (a) Transition function $F(x)$. The area of stability corresponds to the interval $(0,x_1)$, the area of self-organized criticality corresponds to the interval (x_1,x_3), the area of catastrophe corresponds to the interval $(x_3,1)$. (b) Densities of defects vs level of hierarchy for different values of parameter p. Curve (1) corresponds to the area of stability $p=0.1<x_1$; curves (2) and (3) correspond to the area of self-organized criticality $x_1<p<x_3$: $p=0.25$: (2); $p=0.78$: (3); curve (4) corresponds to the area of catastrophe $p=0.92>x_3$. (c) Magnitude-frequency relation for different values of parameter p. (1): in the area of stability $p=0.1<x_1$; (2): in the area of self-organized criticality $x_1<p=0.4<x_3$. A slope of the straight line (2) is equal to unity.

Let us denote three other fixed points as x_1, x_2, and x_3 [Fig. 3(a)]. Points x_1 and x_3 are unstable; intervals $(0,x_1)$ and $(x_3,0)$ correspond to areas of stability and catastrophe, respectively. In contrast to the previous case of phase transition (see Sec. VI), a whole interval (x_1,x_3) exists between these two areas.

In this section we investigate possible kinds of behavior of the densities of defects p_l on the interval (x_1,x_3), when the magnitude-frequency relation is linear on the log/log plot. The linearity of the magnitude-frequency relation for the whole transition interval (x_1,x_3) is referred to as the self-organized criticality phenomenon.

A. The case of stationary behavior

Let us consider a hierarchical system with branch number equal to 6. All configurations containing 2, 5, and 6 defects are critical. The transition function F is as follows:

$$F(x)=15x^2(1-x)^4+6x^5(1-x)+x^6. \tag{10}$$

The function F is plotted on Fig. 3(a). It has five fixed points; three of them—0, x_2, 1—are stable, and the other two—x_1, x_3—are unstable. The interval $(0,x_1)$ of values of the parameter p corresponds to the phase of stability. The densities of defects p_l fall to zero when l grows [Fig. 3(b), curve (1)]; the corresponding magnitude-frequency curve has a double exponential downward bend [Fig. 3(c), curve (1)]. The interval $(x_3,1)$ corresponds to the catastrophic behavior of the system [Fig. 3(b), curve (3)]. If parameter p is inside the interval (x_1,x_3) then the densities of defects p_l tend to the constant value x_2 when l grows [Fig. 3(b), curve (2)].

It follows from Eq. (7) that the corresponding magnitude-frequency relation demonstrates a linear behavior in log/log plot with a slope equal to unity [Fig. 3(c), curve (2)], which is similar to the behavior observed in the critical point of the phase transition from stability to catastrophe [Fig. 2(c), curve (2)].

The linear form of the magnitude-frequency relation exists for all values of parameter p inside the interval (x_1,x_3), so the interval (x_1,x_3) corresponds to the self-organized criticality behavior of the system. The densities of defects tend to a constant value, so this case is referred to as self-organized criticality with a stationary solution.

B. The case of periodic behavior

Let us consider a hierarchical system with branch number 11. All configurations containing 2, 9, 10, and 11 defects are critical. Transition function F is as follows:

$$F(x)=55x^2(1-x)^9+55x^9(1-x)^2+11x^{10}(1-x)+x^{11}. \tag{11}$$

The transition function F has five fixed points [Fig. 4(a)]. There are two stable fixed points (0, 1) and three unstable fixed points (x_1, x_2, and x_3). In contrast to the previous case the absolute value of derivation $F'(x)$ in the fixed point x_2 is greater than unity: $F'(x_2)>1$. The interval $(0,x_1)$ of values of parameters corresponds to the phase of stability [Figs. 4(b), 4(c), curve (1)]. The interval $(x_3,1)$ corresponds to the phase of catastrophe [Fig. 4(b), curve (3)].

If $x_1<p<x_3$ and $p\neq x_2$ then the relation between densities of defects p_l and the corresponding level l is periodic with period equal to 2 [Fig. 4(b), curve (2)]. The magnitude-frequency relation corresponding to the interval (x_1,x_3) of values of the parameter p is linear with a slope equal to unity [Fig. 4(c), curve (2)]. Thus this kind of behavior is referred to as self-organized criticality with a periodic solution.

C. The case of chaotic behavior

Let us consider a hierarchical system with branch number equal to 11. All configurations containing 2, 3, 10, and 11 defects are critical. The transition function F is expressed as follows:

$$F(x)=55x^2(1-x)^9+165x^3(1-x)^8+11x^{10}(1-x)+x^{11}. \tag{12}$$

The transition function F is similar to the one considered in Sec. VII B [Fig. 4(a)]. But the absolute value of the derivation $F'(x)$ in the unstable fixed point x_2 is greater than in previous case. Intervals $(0,x_1)$ and $(x_3,1)$ correspond to the phases of stability and catastrophe, respectively [Fig. 5(a), curves (1,3)].

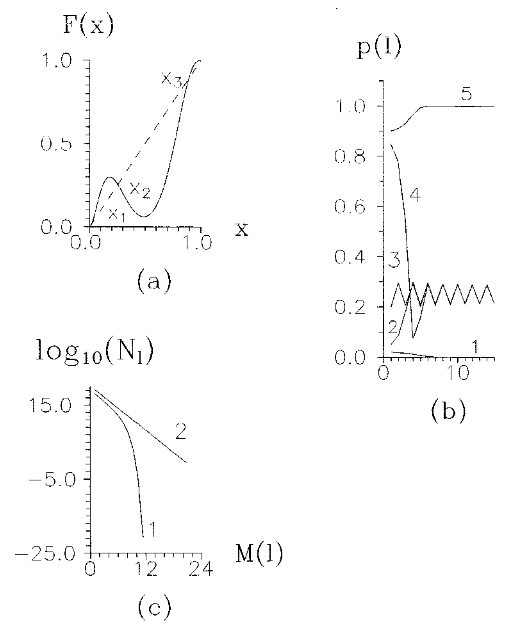

FIG. 4. Hierarchical system representing a periodic behavior in the area of self-organized criticality. Branch number of the model is equal to 11. All configurations containing 2, 9, 10, or 11 defects are critical. (a) Transition function $F(x)$. The area of stability corresponds to the interval $(0,x_1)$, the area of self-organized criticality corresponds to the interval (x_1,x_3), the area of catastrophe corresponds to the interval $(x_3,1)$. (b) Densities of defects vs level of hierarchy for different values of parameter p. Curve (1) corresponds to the area of stability $p=0.02<x_1$; curves (2)–(4) correspond to the area of self-organized criticality $x_1<p<x_3$: $p=0.05$: (2); $p=0.2$: (3); $p=0.85$: (4); curve (5) corresponds to the area of catastrophe $p=0.9>x_3$. (c) Magnitude-frequency relation for different values of parameter p. (1): in the area of stability $p=0.02<x_1$; (2): in the area of self-organized criticality $x_1<p=0.2<x_3$. A slope of the straight line (2) is equal to unity.

If parameter p is inside the interval (x_1,x_3) the relation between densities of defects p_l and the level l is chaotic [Fig. 5(a), curve (2)]. Possible values of densities of defects p_l for big values of level l are represented on Fig. 5(b). The corresponding magnitude-frequency relation is linear with a slope equal to unity [Fig. 5(c), curve (2)], so this kind of system behavior is referred to as self-organized criticality with a chaotic solution.

VIII. THE ALTERNATION OF INTERVALS OF SELF-ORGANIZED CRITICALITY AND CATASTROPHIC BEHAVIOR

In hierarchical systems considered in the previous section the transition function $F(x)$ has 5 fixed points and for the entire interval (x_1,x_3) it satisfies to following condition

$$x_1<F(x)<x_3. \tag{13}$$

Let us consider a system in which the transition function F has 5 fixed points but contradicts the condition (13).

In a hierarchical system with branch number equal to 11 all configurations that contain 2, 3, 4, 5, 8, 9, 10, and 11 defects are critical. Transition function F has the following form:

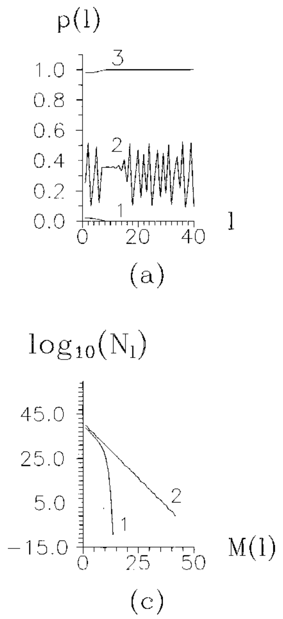

FIG. 5. Hierarchical system representing a chaotic behavior in the area of self-organized criticality. Branch number of the model is equal to 11. All configurations containing 2, 3, 10, or 11 defects are critical. (a) Densities of defects vs level of hierarchy for different values of parameter p. (1): in the area of stability $p=0.02<x_1$; (2): in the area of self-organized criticality $p=0.25$, $x_1<p<x_3$; (3): in the area of catastrophe $p=0.98>x_3$. (b) Possible values of densities p_l for $p=0.25$, levels from $l=4000$ to $l=5000$. (c) Magnitude-frequency relation for different values of parameter p. (1): in the area of stability $p=0.02<x_1$; (2): in the area of self-organized criticality $x_1<p=0.25<x_3$. A slope of the straight line (2) is equal to unity.

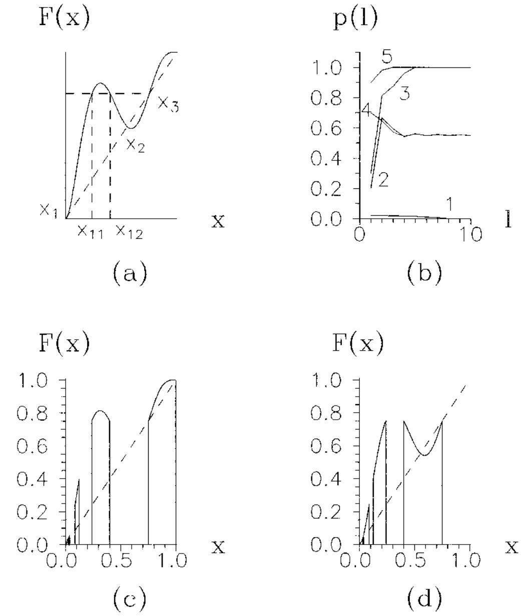

FIG. 6. Hierarchical system representing an alternation of areas of the self-organized criticality and catastrophe. Branch number of the model is equal to 11. All configurations containing 2, 3, 4, 5, 8, 9, 10, or 11 defects are critical. (a) Transition function $F(x)$. Interval $(0,x_1)$ corresponds to the area of stability; intervals (x_{11},x_{12}) and $(x_3,1)$ correspond to the area of catastrophe. (b) Densities of defects vs level of hierarchy for different values of parameter p. (1): in the interval of stability $p=0.02<x_1$; (2): in the interval of self-organized criticality $p=0.2$; (3): in the interval of catastrophe $x_{11}<p=0.3<x_{12}$; (4): in the interval of self-organized criticality $p=0.7$; (5): in the interval of catastrophe $p=0.9>x_3$. (c) Areas of catastrophe. (d) Areas of self-organized criticality.

$$F(x)=55x^2(1-x)^9+165x^3(1-x)^8+330x^4(1-x)^5 +462x^5(1-x)^6+165x^8(1-x)^3+55x^9(1-x)^2 +11x^{10}(1-x)+x^{11}. \tag{14}$$

Maximal value of the function $F(x)$ for the interval (x_1,x_2) is greater than x_3 [Fig. 6(a)]. If parameter p is chosen inside the interval (x_1,x_2), so that $F(p)>x_3$, then the densities of defects p_l tend to unity when l grows [Fig. 6(b), curve (3)]. Thus the system demonstrates the catastrophic behavior not only for the interval $(x_3,1)$ but also for the infinite sequence of intervals Δx_i. For each interval Δx_i there is an integer number i such that the composition $F^{(i)}$ applied to the interval Δx_i maps the interval into the interval $(x_3,1)$. Intervals Δx_i may be constructed as a set of images of the interval $\Delta x_1=(x_{11},x_{12})$ for the composition F^{-i}. The initial interval (x_{11},x_{12}) is defined as follows:

$$x_1<x_{11}<x_{12}<x_2, \quad F(x_{11})=F(x_{12})=x_3. \tag{15}$$

If parameter p is chosen inside the interval (x_1,x_3) but outside intervals Δx_i then the densities of defects p_l tend to the constant value x_2. Thus, the set of intervals additional to the intervals Δx_i correspond to self-organized criticality [Fig. 6(b), curves (2) and (4)]. The areas of catastrophic and self-organized criticality behavior are represented in Figs. 6(c) and 6(d), respectively.

IX. THE ALTERNATION OF INTERVALS OF STABILITY AND SELF-ORGANIZED CRITICALITY BEHAVIOR

In this section we propose an example of a system with alternating intervals of stability and self-organized criticality behavior. Let us consider a hierarchical system with branch number 15. All configurations containing 2, 14, and 15 defects are critical. The transition function $F(x)$ is as follows:

$$F(x)=105x^2(1-x)^{13}+15x^{14}(1-x)+x^{15}. \tag{16}$$

The minimum value of function $F(x)$ on the interval (x_2,x_3) is approximately equal to 2.5×10^{-3}, which is less than $x_1\approx1.1\times10^{-2}$ [Figs. 7(a), 7(b)]. Then there are areas of stability inside the interval (x_2,x_3) that alternate with areas of self-organized criticality behavior.

This alternation is similar to that described in Sec. VIII. If parameter p of the system is inside an area of stability, the densities of defects p_l tend to zero when the level l grows [Fig. 7(c), curves (1) and (3)]. If parameter p is inside an area of self-organized criticality, then the densities p_l tend to a periodic solution with period equal to 4 [Fig. 7(c), curves (2) and (4)]. If parameter p is greater than x_3, than the densities tend to unity, which corresponds to catastrophic behavior [Fig. 7(b), curve (5)].

It is possible to obtain a more complicated behavior including the alternation of intervals of stability, self-organized criticality, and catastrophe inside the interval x_1,x_3. The

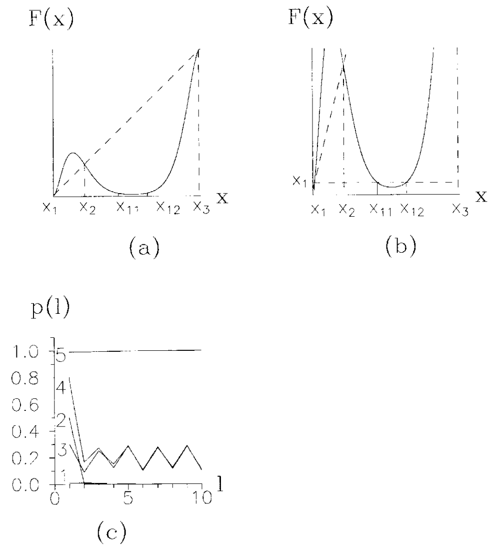

FIG. 7. Hierarchical system representing an alternation of areas of stability and self-organized criticality. Branch number of the model is equal to 15. All configurations containing 2, 14, or 15 defects are critical. (a) Transition function $F(x)$. Intervals $(0,x_1)$ and (x_{11},x_{12}) correspond to the area of stability, interval $(x_3,1)$ corresponds to the area of catastrophe. (b) The enlargement of (a) for $0<F(x)<0.1$. (c) Densities of defects vs level of hierarchy for different values of parameter p. (1): in the interval of stability $p=0.01<x_1$; (2): in the interval of self-organized criticality $p=0.3$; (3): in the interval of stability $x_{11}<p=0.5<x_{12}$; (4): in the interval of self-organized criticality $p=0.8$; (5): in the interval of catastrophe $p=0.991>x_3$.

transition function $F(x)$ which determines this kind of alternation inside the interval x_1,x_3, satisfies the following conditions:

$$\min_{(x_1,x_3)} F(x)<x_1, \tag{17}$$

$$\max_{(x_1,x_3)} F(x)>x_3. \tag{18}$$

We do not consider examples of this kind of system behavior because it is a composition of simple behaviors considered above.

X. DISCUSSION OF RESULTS

We investigated the basic kinds of transition behavior from stability to catastrophe for a simple hierarchical system of defect development. The transition function F determines the behavior of the system. Hierarchical systems with three fixed points of the transition function F demonstrate a phase transition from stability to catastrophe when fixed points $F(0)=0$ and $F(1)=1$ are stable. It is claimed that if the transition function F has five fixed points, then instead of a phase transition point, a transition interval appears between the areas of stability and catastrophe. The behavior of the system inside the transition interval (x_1,x_3) was investigated.

It is observed that an area of the self-organized criticality always exists inside the transition interval (x_1,x_3). The area of self-organized criticality coincides with the transition interval if the transition function F satisfies to the following conditions:

$$\min_{(x_1,x_3)} F(x)\geqslant x_1, \tag{19}$$

$$\max_{(x_1,x_3)} F(x)\leqslant x_3. \tag{20}$$

If the transition function contradicts to the condition (19), then the alternation of areas of self-organized criticality with areas of stability appears inside the transition interval (x_1,x_3). If the transition function F contradicts the condition (20), then the alternation of areas of self-organized criticality with areas of catastrophe appears. The alternation of areas of all possible kinds of system behavior — stability, self-organized criticality, and catastrophe — appears inside the transition interval (x_1,x_3), when the transition function F satisfies both conditions (17) and (18).

The alternation of areas of self-organized criticality with areas of catastrophe and/or stability is a new interesting feature of the described class of hierarchical systems. The size of alternating areas tends to zero in a neighborhood of the corresponding fixed point of the transition function $F(x)$ [Figs. 6(c) and 6(d)]. It means that in the neighborhood of this point the observed behavior became unstable and a small change of system parameter leads to multiple changes of system behavior. There is an infinite number of phase transitions from self-organized criticality to catastrophe in the neighborhood of the fixed point x_1 [Figs. 6(c) and 6(d)]. There is some similarity with a cascade of phase transitions observed in the theory of spin glasses [22].

The heterogeneity of space distribution of earthquakes may be connected with an alternation of self-organized criticality and stability areas. If small variations of parameter, as we obtained in the model, lead to the change of observed behavior then the small spatial variations of the stress in lithosphere will naturally lead to the spatial heterogeneity of seismicity.

The self-organized criticality corresponds to different kinds of relations between the densities of defects p_l and level l. The behavior of densities of defects may be stable, periodic, and chaotic. It depends on the derivation of the transition function F in the fixed point x_2. The magnitude-frequency relation in all cases remains linear with small deviations from the main trend. Some deviations of the magnitude-frequency relation from the linearity in seismic observations may be connected with a chaotic dependence of the number of earthquakes on their sizes, exhibiting the transition area of self-organized criticality.

We did not investigate all kinds of possible transition behavior from stability to catastrophe. To obtain more complicated behavior in a similar construction it is possible to take a larger branch number n and construct a transition function $F(x)$ with more fixed points or stronger nonmonotone behavior.

The suggested hierarchical constructions reflect general properties of the self-organized criticality phenomenon and cannot be applied for a more detailed analysis of a particular physical system. Nevertheless the nontrivial and complex behavior of simple systems described above demonstrates that hierarchical systems are interesting objects of future investigations.

ACKNOWLEDGMENTS

This work was supported by the U.S. National Science Foundation Grant No. EAR-94-23818 and International Science and Technology Center Project No. 008-94.

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